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A PATH-INDEPENDENT INTEGRAL FOR TRANSIENT CRACK PROBLEMS

FRED NILSSON

Department of Strength of Materials and Solid Mechanics,
The Royal Institute of Technology, Stockholm

Abstract—The problem of a stationary crack in a visco-elastic body under plane strain, subjected to dynamic loading, is formulated within the realms of the classical theory. By introducing the Laplace-transform with respect to time the problem is reformulated in the transformed space. A line-integral is defined and it is shown that it is path-independent. The relation between this integral and the transformed stress-intensity factor is derived. It is indicated that the integral may be valuable for calculations of dynamic stress-intensity factors. In order to illustrate this, a simple example is solved and some numerical results are presented.

INTRODUCTION

A USEFUL tool in fracture analysis is the path-independent J-integral proposed by Rice [1]. Beside its possible use as a criterion for crack growth (cf. [2]), it has proved to be a valuable tool in determinations of stress-intensity factors. This is a consequence of the property of path-independence which implies that information about the state in the vicinity of the crack tip can be obtained by studying the far away field. However, the use of the J-integral is limited to static cases where inertia effects can be neglected.

When high loading rates are present the inertia effects may play an important role. The object of this paper is to derive a concept similar to the J-integral which can be applied to such situations. We will limit ourselves to the transient response of bodies containing a stationary crack. Of course it would be of interest to extend the concept to transient problems of moving cracks, but this seems to be very difficult, if at all possible (cf. [3]).

The integral concept discussed in this paper is not defined in the space of the original variables, but instead in the space of a Laplace transform of the time variable. This excludes the possibility of treating nonlinear elastic materials. On the other hand the method enables us to handle cases for linearly visco-elastic materials.

GOVERNING EQUATIONS AND THE LAPLACE TRANSFORM

Consider a plate under plane strain, mode I conditions. The analysis applies equally well to mode II and III cases, but these are omitted for brevity. The plate contains a traction-free straight crack along the x_1-axis (Fig. 1).

The material is assumed to be isotropic and linearly visco-elastic, thus obeying the constitutive relations

$$s_{ij} = \int_0^t G_1(t-\tau)\frac{\mathrm{d}e_{ij}(\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau \tag{1}$$

$$\sigma_{kk} = \int_0^t G_2(t-\tau)\frac{\mathrm{d}\varepsilon_{kk}(\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau \tag{2}$$

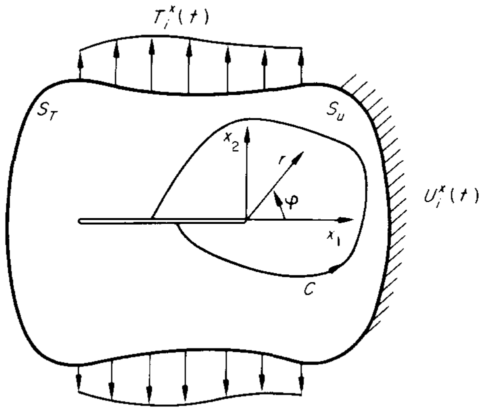

FIG. 1. The plane crack-problem.

s_{ij} is the stress deviation tensor

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij} \tag{3}$$

and e_{ij} the strain deviation tensor

$$e_{ij} = \varepsilon_{ij} - \tfrac{1}{3}\varepsilon_{kk}\delta_{ij} \tag{4}$$

$G_1(t)$ is the relaxation-modulus for pure shear and $G_2(t)$ is relaxation modulus for pure compression.

The deformation is governed by the equations of motion

$$\sigma_{ij,j} = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{5}$$

where ρ is the material density, assumed to be a constant. u_i is the displacement vector, where $u_3 \equiv 0$ due to the plane strain assumption. Furthermore compatibility is ensured if

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{6}$$

Finally the boundary conditions can be stated as

$$\sigma_{ij}n_j = T_i^* \quad \text{on } S_T \tag{7}$$

$$u_i = u_i^* \quad \text{on } S_u \tag{8}$$

n_j is the normal vector of a surface element, T_i^* are the prescribed tractions on the surface S_T and u_i^* are the prescribed displacements on the surface S_u. In the above equations all quantities may vary with time except n_j, ρ, S_T and S_u.

A standard procedure in the analysis of dynamic problems in linear visco–elasticity is to introduce the Laplace transform

$$f(p) = \int_0^\infty f(t)\,\mathrm{e}^{-pt}\,\mathrm{d}t. \tag{9}$$

Assume the initial conditions to be zero. Since S_T and S_u are time-independent, it is possible to apply the Laplace transform to equations (1–8). This yields

$$\bar{s}_{ij} = p\bar{G}_1\bar{e}_{ij} \tag{10}$$

$$\bar{\sigma}_{kk} = p\bar{G}_2\bar{\varepsilon}_{kk} \tag{11}$$

$$\bar{s}_{ij} = \bar{\sigma}_{ij}-\tfrac{1}{3}\bar{\sigma}_{kk}\delta_{ij} \tag{12}$$

$$\bar{e}_{ij} = \bar{\varepsilon}_{ij}-\tfrac{1}{3}\bar{\varepsilon}_{kk}\delta_{ij} \tag{13}$$

$$\bar{\sigma}_{ij,j} = \rho p^2\bar{u}_i \tag{14}$$

$$\bar{\varepsilon}_{ij} = \tfrac{1}{2}(\bar{u}_{i,j}+\bar{u}_{j,i}) \tag{15}$$

$$\bar{\sigma}_{ij}n_j = \bar{T}_i^* \quad \text{on } S_T \tag{16}$$

$$\bar{u}_i = \bar{u}_i^* \quad \text{on } S_u. \tag{17}$$

The transformed equations are similar in structure to those for a static linearly elastic problem. We can interpret this new set of equations as an elastic problem with body forces proportional to u_i.

It is convenient to define a functional $\bar{V}(\bar{\varepsilon}_{ij})$ analogous to the ordinary strain energy density

$$\bar{V}(\bar{\varepsilon}_{ij}) = \tfrac{1}{2}\bar{\sigma}_{ij}\bar{\varepsilon}_{ij}. \tag{18}$$

It then follows from the linearity of the transformed constitutive relations equations (10–13) that

$$\bar{\sigma}_{ij} = \frac{\partial\bar{V}}{\partial\bar{\varepsilon}_{ij}}. \tag{19}$$

Note that the inverse of $\bar{V}(\bar{\varepsilon}_{ij})$ is not equal to the strain-energy density, but has a more complicated physical interpretation.

THE PATH-INDEPENDENT INTEGRAL

Guided by the static case, let us consider the following integral along a path C embracing the crack-tip (Fig. 1)

$$\bar{I} = \int_C (\bar{V}+\tfrac{1}{2}\rho p^2\bar{u}_i\bar{u}_i)\,\mathrm{d}x_2 - \bar{T}_i\frac{\partial\bar{u}_i}{\partial x_1}\,\mathrm{d}s. \tag{20}$$

It will now be shown that $\bar{I}$ is independent of the choice of C. It is sufficient to show that $\bar{I}$ vanishes for any closed curve C^*, since $\mathrm{d}x_2$ and the transformed tractions $\bar{T}_i$ are zero along the crack surface (cf. [1]). Thus let C^* form a closed curve and let A^* denote the enclosed area. Using the relation

$$\bar{\sigma}_{ij}n_j = \bar{T}_i \tag{21}$$

and Gauss' theorem, equation (20) can be transformed into the surface integral

$$\bar{I}_{C^*} = \int_{A^*}\left[\frac{\partial\bar{V}}{\partial x_1}+\tfrac{1}{2}\rho p^2\frac{\partial}{\partial x_1}(\bar{u}_i\bar{u}_i)-\frac{\partial}{\partial x_j}\left(\bar{\sigma}_{ij}\frac{\partial\bar{u}_i}{\partial x_1}\right)\right]\mathrm{d}A. \tag{22}$$

But from equation (19)

$$\frac{\partial \bar{V}}{\partial x_1} = \frac{\partial \bar{V}}{\partial \bar{\varepsilon}_{ij}}\frac{\partial \bar{\varepsilon}_{ij}}{\partial x_1} = \bar{\sigma}_{ij}\frac{\partial}{\partial x_1}\left(\frac{\partial \bar{u}_i}{\partial x_j}\right) \tag{23}$$

where equation (15) and the fact that $\bar{\sigma}_{ij}$ and $\bar{\varepsilon}_{ij}$ are symmetric, have been used.

Insertion of equation (23) into equation (22) and some trivial manipulations yield

$$\bar{I}_{C^*} = \int_{A^*} \frac{\partial \bar{u}_i}{\partial x_1}\left(\rho p^2 \bar{u}_i - \frac{\partial \bar{\sigma}_{ij}}{\partial x_j}\right) \mathrm{d}A \tag{24}$$

which vanishes on account of the transformed equations of motion (14). Thus $\bar{I}$ is zero for any closed contour C^* and it follows that $\bar{I}$ is path-independent for any curve C embracing the crack-tip.

RELATION BETWEEN I AND THE STRESS-INTENSITY FACTOR

We have above established that the integral $\bar{I}$ is path-independent. To use it for estimates of stress-intensity factors, we need a relation between the two. To this end let us consider the near crack-tip field. It can be shown [4], that for elastic problems the singularity has the same form for transient crack problems as for the static ones. Thus in a polar coordinate system we have in the elastic case

$$\sigma_{ij} = \frac{K_I(t)}{\sqrt{2\pi r}} f_{ij}(\varphi) \qquad \text{as } r \to 0 \tag{25}$$

$$u_i = \frac{K_I(t)}{\mu}\sqrt{\frac{r}{2\pi}}\, g_i(\varphi, \nu) \qquad \text{as } r \to 0. \tag{26}$$

Here μ is the shear modulus and ν Poisson's ratio. Performing the Laplace transform we obtain for the purely elastic case

$$\bar{\sigma}_{ij} = \frac{\bar{K}_I(p)}{\sqrt{2\pi r}} f_{ij}(\varphi) \qquad \text{as } r \to 0 \tag{27}$$

$$\bar{u}_i = \frac{\bar{K}_I(p)}{\mu}\sqrt{\frac{2r}{\pi}}\, g_i(\varphi, \nu) \qquad \text{as } r \to 0. \tag{28}$$

Now evaluate $\bar{I}$ along a circular path centered at the crack-tip and with radius R. Then

$$\bar{I} = R\int_{\varphi=-\pi}^{\pi} \left[(\tfrac{1}{2}\bar{\sigma}_{ij}\bar{\varepsilon}_{ij} + \tfrac{1}{2}\rho p^2 \bar{u}_i \bar{u}_i)\cos\varphi - \bar{\sigma}_{ij} n_j \frac{\partial \bar{u}_i}{\partial x_1}\right] \mathrm{d}\varphi. \tag{29}$$

From equation (29) we see that only terms of order R^{-1} will give a finite contribution to $\bar{I}$ as $R \to 0$. The first and the third term are both of order $\bar{\sigma}_{ij}\bar{\varepsilon}_{ij} \sim R^{-1}$ according to equation (27). The second term is of order $\bar{u}_i\bar{u}_i \sim R$ and will thus give zero contribution as $R \to 0$. Then equation (29) can be written

$$\bar{I} = R\int_{\varphi=-\pi}^{\pi} \left(\tfrac{1}{2}\bar{\sigma}_{ij}\bar{\varepsilon}_{ij}\cos\varphi - \bar{\sigma}_{ij} n_j \frac{\partial \bar{u}_i}{\partial x_1}\right) \mathrm{d}\varphi. \tag{30}$$

But since the singular parts of $\bar{\sigma}_{ij}$ and $\bar{\varepsilon}_{ij}$ have exactly the same form as σ_{ij}, ε_{ij} in the static case, the value of $\bar{I}$ takes the well-known form

$$\bar{I} = \bar{K}_{\mathrm{I}}^2(p)\frac{1-\nu}{2\mu} = \bar{K}_{\mathrm{I}}^2(p)\frac{1-\nu^2}{E} \qquad \text{plane strain} \tag{31}$$

where E is Young's modulus. Similar arguments give

$$\bar{I} = \bar{K}_{\mathrm{I}}^2(p)\frac{1}{E} \qquad \text{plane stress} \tag{32}$$

$$\bar{I} = \bar{K}_{\mathrm{III}}^2\frac{1}{2\mu} \qquad \text{anti-plane strain.} \tag{33}$$

Now turning to the visco–elastic case, it is seen from the correspondence principle expressed by equations (10–13) that the above derivation remains valid if we make suitable substitutions for μ and ν into equations (26) and (31–33). Considering equations (10–13) and using well-known relations between the elastic constants, it is found that

$$\mu \to \frac{p\bar{G}_1}{2} \tag{34}$$

$$\nu \to \frac{\bar{G}_2-\bar{G}_1}{2\bar{G}_2+\bar{G}_1}. \tag{35}$$

Thus in the visco–elastic case we have

$$\bar{I} = \bar{K}_{\mathrm{I}}^2(p)\frac{\bar{G}_2+2\bar{G}_1}{p\bar{G}_1(2\bar{G}_2+\bar{G}_1)} \qquad \text{plane strain} \tag{36}$$

$$\bar{I} = \bar{K}_{\mathrm{I}}^2(p)\frac{2G_2+G_1}{3p\bar{G}_1\bar{G}_2} \qquad \text{plane stress} \tag{37}$$

$$\bar{I} = \bar{K}_{\mathrm{III}}^2(p)\frac{1}{p\bar{G}_1} \qquad \text{anti-plane strain.} \tag{38}$$

It must be pointed out that the inverse of $\bar{I}$ is not equal to the energy-release rate, instead more complicated relations will result. The value of $\bar{I}$ lies in its computational usefulness as will be shown by some examples below.

THE INFINITE STRIP PROBLEM

Consider a strip of purely elastic material under conditions of plane strain situated between $x_2 = \pm h$ (Fig. 2). The strip contains a semi-infinite crack along the x_1-axis from $x_1 = -\infty$ to $x_1 = 0$. The boundary conditions are given by

$$u_1 = 0, \qquad u_2 = \pm u_0 q(t) \quad \text{on } x_2 = \pm h \tag{39}$$

$$\sigma_{22} = \sigma_{12} = 0 \quad \text{on } x_2 = 0, x_1 < 0 \tag{40}$$

u_0 is a constant and $q(t)$ a dimensionless function of time, which is zero for $t < 0$.

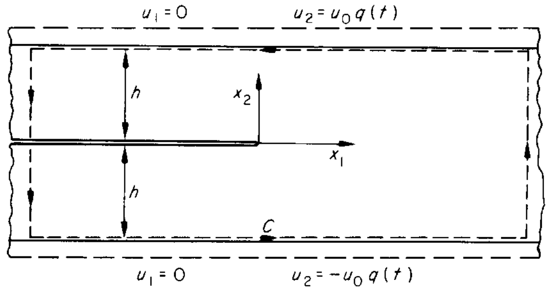

FIG. 2. The infinite strip.

The motion of an elastic body under plane strain is governed by the wave-equations

$$\Phi_{,11} + \Phi_{,22} = \frac{1}{C_1^2}\Phi_{,tt} \tag{41}$$

$$\Psi_{,11} + \Psi_{,22} = \frac{1}{C_2^2}\Psi_{,tt} \tag{42}$$

where

$$u_1 = \Phi_{,1} - \Psi_{,2} \tag{43}$$

$$u_2 = \Phi_{,2} + \Psi_{,1} \tag{44}$$

and C_1 is longitudinal wave velocity and C_2 the shear-wave velocity.

Taking the Laplace-transform of equations (41–44) and calculating the transformed stresses give

$$\bar{\Phi}_{,11} + \bar{\Phi}_{,22} = \frac{p^2}{C_1^2}\bar{\Phi} \tag{45}$$

$$\bar{\Psi}_{,11} + \bar{\Psi}_{,22} = \frac{p^2}{C_2^2}\bar{\Psi} \tag{46}$$

$$\bar{u}_1 = \bar{\Phi}_{,1} - \bar{\Psi}_{,2} \tag{47}$$

$$\bar{u}_2 = \bar{\Phi}_{,2} + \bar{\Psi}_{,1} \tag{48}$$

$$\bar{\sigma}_{11} = 2\mu\left[\bar{\Phi}_{,11} - \bar{\Psi}_{,12} + \frac{\nu}{1-2\nu}(\bar{\Phi}_{,11} + \bar{\Phi}_{,22})\right] \tag{49}$$

$$\bar{\sigma}_{22} = 2\mu\left[\bar{\Phi}_{,22} + \bar{\Psi}_{,12} + \frac{\nu}{1-2\nu}(\Phi_{,11} + \bar{\Phi}_{,22})\right] \tag{50}$$

$$\bar{\sigma}_{12} = \mu(2\bar{\Phi}_{,12} - \bar{\Psi}_{,22} + \bar{\Psi}_{,11}). \tag{51}$$

We now wish to evaluate $\bar{I}$ along the path sketched in Fig. 2. It is immediately seen from the boundary-conditions and the definition of $\bar{I}$ that the only non-zero contributions come from the lines crossing the strip. Furthermore because of the elliptic character of equations (45) and (46) and the type of boundary conditions, derivatives with respect to x_1 vanish as $x_1 \to \pm\infty$. The path-independence of $\bar{I}$ permits us to extend the integration path, so that it crosses the strip infinitely far from the crack-tip where equations (45) and (46) then take the form

$$\bar{\Phi}_{,22} = \frac{p^2}{C_1^2}\bar{\Phi} \tag{52}$$

$$\bar{\Psi}_{,22} = \frac{p^2}{C_2^2}\bar{\Psi} \tag{53}$$

with the solutions

$$\bar{\Phi} = A_1\,\mathrm{e}^{(p/C_1)x_2} + B_1\,\mathrm{e}^{-(p/C_1)x_2} \tag{54}$$

$$\bar{\Psi} = A_2\,\mathrm{e}^{(p/C_2)x_2} + B_2\,\mathrm{e}^{-(p/C_2)x_2}. \tag{55}$$

For $x_1 \to \infty$ we have the boundary-conditions

$$x_2 = h : \bar{u}_1 = 0 \tag{56}$$

$$\bar{u}_2 = u_0\bar{q}(p) \tag{57}$$

$$x_2 = 0 : \bar{u}_2 = 0 \tag{58}$$

$$\bar{\sigma}_{12} = 0. \tag{59}$$

Insertion of equations (47), (48), (54) and (55) into equations (56–59) yields after some trivial calculations

$$x_1 \to +\infty : \bar{u}_2 = u_0\bar{q}(p)\frac{\sinh(px_2/C_1)}{\sinh(ph/C_1)} \tag{60}$$

$$\bar{\varepsilon}_{22} = \frac{1}{E}\bar{\sigma}_{22}\frac{1-\nu-2\nu^2}{1-\nu} = u_0\bar{q}(p)\frac{p}{C_1}\frac{\cosh(px_2/C_1)}{\sinh(ph/C_1)}\,. \tag{61}$$

For $x_1 \to -\infty$ the boundary-condition equation (58) is replaced by

$$\bar{\sigma}_{22} = 0. \tag{62}$$

The resulting solution becomes in an analogous way

$$x_1 \to -\infty : \bar{u}_2 = u_0\bar{q}(p)\frac{\cosh(px_2/C_1)}{\cosh(ph/C_1)}$$

$$\bar{\varepsilon}_{22} = \frac{1}{E}\bar{\sigma}_{22}\frac{1-\nu-2\nu^2}{1-\nu} = u_0\bar{q}(p)\frac{p}{C_1}\frac{\sinh(px_2/C_1)}{\cosh(ph/C_1)}. \tag{63}$$

The other components of $\bar{u}_i$, $\bar{\varepsilon}_{ij}$ and $\bar{\sigma}_{ij}$ are either zero or do not give any contribution to the $\bar{I}$-integral.

Inserting the expressions equations (60–63) into equation (20) and performing the integrations yield

$$\bar{I} = u_0^2\bar{q}^2(p)p\rho\frac{2C_1}{\sinh(2ph/C_1)}. \tag{64}$$

Equation (31) gives with a little manipulation the following result for the transform of the stress-intensity factor

$$\bar{K}_I(p) = \frac{u_0 E}{(h(1+\nu)^2(1-2\nu))^{\frac{1}{2}}}\bar{q}(p)\left[\frac{2ph/C_1}{\sinh(2ph/C_1)}\right]^{\frac{1}{2}}. \tag{65}$$

This expression can then be inverted when the form of $q(t)$ is given. Take as an example a linearly increasing time-function

$$q(t) = \frac{t}{t_0} \tag{66}$$

$$\bar{q}(p) = \frac{1}{t_0 p^2}. \tag{67}$$

The inverse of equation (65) can in this case readily be calculated by the use of an integral transform table, Ref. [5]

$$K_I(t) = \frac{u_0 E}{(h(1+\nu)^2(1-2\nu))^{\frac{1}{2}}}\cdot\frac{h}{t_0 C_1}\cdot\frac{4}{\sqrt{\pi}}\cdot\sum_{0\leq n\leq(C_1 t/4h)}(-1)^n\binom{-\frac{1}{2}}{n}\left(\frac{C_1 t}{h}-1-4n\right)^{\frac{1}{2}}. \tag{68}$$

Here n is an integer and $\binom{-\frac{1}{2}}{n}$ denotes generalized binomial coefficients. Equation (68) is plotted in Fig. 3. Note that $K_I(t)$ in Fig. 3 has been normalized with respect to the value that would result from a quasi-static analysis

$$K_{I_{qs}} = \frac{u_0 E}{(h(1+\nu)^2(1-2\nu))^{\frac{1}{2}}}\cdot\frac{t}{t_0}. \tag{69}$$

It is seen that $K_I(t)$ oscillates around the quasi-static value and that as time increases the quotient rapidly approaches unity.

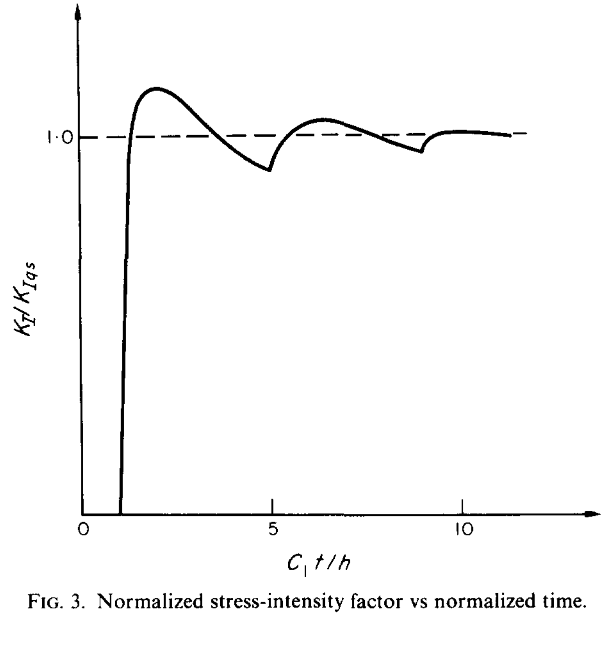

FIG. 3. Normalized stress-intensity factor vs normalized time.

DISCUSSION

In the problem considered, the exact solution could be derived rather easily. In general this is not possible even with this method. However, the present method is a very powerful tool for approximate or numerical analysis of crack problems. Through the Laplace-transform a transient dynamic problem is converted to a static problem in the Laplace-space with body forces proportional to the displacement vector. This problem can easily be solved using e.g. a finite element method procedure for suitable values of p. Especially $\bar{I}(p)$ can be determined. Employing some numerical inversion technique $\bar{I}(p)$ can then be inverted and thus the stress intensity factor determined as a function of time. The value of the $\bar{I}$-concept is now obvious. It is often, especially with the FEM-method difficult to get an accurate description of the field near the crack tip. The far field can be calculated with higher accuracy and by using a suitable large radius path for $\bar{I}$ a good result can be expected. The method of solution presented here is obviously much simpler than exact solution of the wave-equations directly in the original variables.

Even when an analytical solution of the transformed equations is attempted, it may be easier to derive the far-field than the near crack-tip field. This is the case in the example presented above. In fact it was then possible to obtain the far-field without actually solving the boundary value problem.

This suggests that in cases where it is not possible to solve the far-field exactly, a more or less simple approximation may lead to a good estimate of $K(t)$. Of course the accuracy of this estimate will depend heavily upon the type of problem considered and the form of the approximating functions.

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Абстракт—В области классической теории определяются формулы для задачи стационарной трещины в вязкоупругом теле, для случая плоской деформации, под влиянием динамической нагрузки. Путем введения преобразования по Лапласу в зависимости от времени, дается новая формулировка задачи в преобразованном пространстве. Определяется линейный интеграл, который не зависит от траектории. Дается зависимость между этим интегралом и преобразованным фактором интенсивности напряжений. Указывается на значение этого интеграла для расчетов факторов интенсивности динамических напряжений. С целью иллюстрации этого, подсчитан несложный пример и даются некоторые численные результаты.